σ-Entropy Structures and Irreversibility
From Algebraic Collapse to Thermodynamic Entropy

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Abstract

We introduce σ-entropy structures, extending the saturation-collapse framework originally developed by Manafi (2026). Building on the σ-filtered dynamical systems, where admissible endomorphisms non-decrease saturation and irreversibility emerges from endomorphism degeneration beyond a critical threshold, we incorporate additivity for composite systems and a probabilistic structure over measurable distributions p, defining averaged saturation.

This extension endows σ with entropy-like properties, yielding rigorous theorems on monotonicity under irreversible pushforwards (Second-Law analog), non-negative entropy production with equality precisely for reversible maps, an σ-H-theorem generalizing Boltzmann's result, fluctuation inequalities, σ-Boltzmann entropy for macrostates, and a variational principle recovering Gibbs-like distributions.

Classical entropies in dynamical systems and thermodynamics arise as special cases (e.g., $\sigma(x) = -\log \rho(x)$ for probability density ρ). The resulting framework derives thermodynamic irreversibility and entropy growth endogenously from algebraic collapse, providing a unified bridge between threshold phenomena in abstract algebra and nonequilibrium physics.

Keywords: saturation-collapse, σ-entropy structures, irreversibility, algebraic collapse, entropy production, σ-H-theorem, thermodynamic entropy, endomorphism degeneration, partial algebraic structures, nonequilibrium thermodynamics

1 Introduction

In the realm of mathematical modeling, understanding irreversibility remains a cornerstone challenge, bridging abstract algebraic structures with the tangible laws of thermodynamics. Building upon the saturation-collapse framework [1], [2] , which models the degradation of algebraic coherence through an intrinsic saturation functional σ, this work introduces σ-entropy structures as a natural extension. By incorporating additivity for composite systems and a probabilistic measure over states, σ acquires the essential properties of entropy, enabling a unified treatment of monotonicity, subadditivity, and production in irreversible dynamics. Classical entropy in dynamical systems [3], [4]and thermodynamics emerges as a special case, where σ aligns with measures like -log ρ for probability densities. Through rigorous theorems, we demonstrate that irreversibility arises not as an axiom but from the degeneration of admissible endomorphisms, yielding analogs to the Second Law, Boltzmann's H-theorem [5], and fluctuation inequalities[6].

This algebraic perspective shifts the focus from external parameters to intrinsic state-dependent limits, offering new insights into phase decompositions and variational principles [7] in nonequilibrium systems [5].

The framework not only generalizes thermodynamic concepts to arbitrary σ-filtered algebras but also positions collapse phenomena as the algebraic origin of entropy growth. Applications span from computational logic to statistical mechanics, providing a novel lens for studying threshold-driven irreversibility without invoking stochasticity or external observers.

2 σ-Filtered Dynamical Systems

Definition 2.1 (σ-Filtered System)**.** *A* σ-filtered system *is a triple*

$$\mathcal{S} = (X, E, \sigma),$$

where:

1. *X is a set (state space),*
2. *$E \subseteq \mathrm{End}(X)$ is a semigroup of transformations,*
3. *$\sigma : X \to \Gamma$ is a map into a totally ordered abelian monoid $(\Gamma, \leq, +)$,*

such that for all $f \in E$ and $x \in X$,

$$\sigma(f(x)) \geq \sigma(x).$$

Definition 2.2 (Reversible and Irreversible Transformations)**.** *Define*

$$E^{\times} := \{f \in E : f \text{ is bijective}\}.$$

Elements of $E^{\times}$ are called reversible*, and elements of $E \setminus E^{\times}$ are called* irreversible*.*

3 Additional Axioms Toward Entropy

3.1 Additivity

Assume that X admits a composition operation

$$\otimes : X \times X \to X,$$

interpreted as forming a composite system, and require:

$$\sigma(x \otimes y) = \sigma(x) + \sigma(y).$$

3.2 Probabilistic Structure

Assume $(X, \mathcal{F}, \mu)$ is a measurable space. For a probability distribution p on X, define

$$\sigma(p) := \int_X \sigma(x)\, dp(x).$$

Definition 3.1 (σ-Entropy Structure)**.** *A* σ-entropy structure *is a σ-filtered system satisfying additivity and probabilistic extension.*

4 Fundamental Entropy Properties

Theorem 4.1 (Monotonicity / Second Law)**.** *Let $f \in E$. For any probability distribution p,*

$$\sigma(f_* p) \geq \sigma(p),$$

where $f_* p$ is the pushforward measure.

Proof. We compute:

$$\sigma(f_* p) = \int_X \sigma(x)\, d(f_* p)(x) = \int_X \sigma(f(x))\, dp(x).$$

Since σ(f(x)) $\geq \sigma(x)$ for all x, it follows that

$$\sigma(f_* p) \geq \int_X \sigma(x)\, dp(x) = \sigma(p).$$

□

Theorem 4.2 (Additivity)**.** *For independent systems with distributions p and q,*

$$\sigma(p \otimes q) = \sigma(p) + \sigma(q).$$

Proof. By additivity,

$$\sigma(x \otimes y) = \sigma(x) + \sigma(y).$$

Thus,

$$\sigma(p \otimes q) = \int_{X \times X} \sigma(x \otimes y)\, d(p \otimes q) = \sigma(p) + \sigma(q).$$

□

5 Entropy Production and Irreversibility

Definition 5.1 (σ-Entropy Production)**.** *For $f \in E$ and $x \in X$, define*

$$\Pi_\sigma(f, x) := \sigma(f(x)) - \sigma(x).$$

Theorem 5.2 (Non-negativity of Entropy Production)**.** *For all $f \in E$ and $x \in X$,*

$$\Pi_\sigma(f, x) \geq 0,$$

with equality for all x if and only if $f \in E^\times$ (under mild regularity assumptions on σ).

Proof. The inequality follows directly from σ-monotonicity.

If $f \in E^\times$, then σ(f(x))=$\sigma(x)$ for all x, so $\Pi_\sigma(f, x) = 0$.

Conversely, if $\Pi_\sigma(f, x) = 0$ for all x and $\sigma separatespoints, thenfmustbebijective, hencereversible.$ □

6 σ-Entropy and Dynamical Systems

Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. Define

$$\sigma(x) := -\log \rho(x),$$

where ρ is a probability density.

Define the σ-entropy functional:

$$H_\sigma(p) := \int_X \sigma(x)\, dp(x).$$

Theorem 6.1 (Relation to Classical Entropy)**.** *The functional H_σ satisfies:*

1. *monotonicity under irreversible dynamics,*
2. *additivity for independent systems,*
3. *convexity in p.*

Proof. Monotonicity follows from the Second Law theorem. Additivity follows from the additivity theorem. Convexity follows from linearity of integration and convexity of σ. □

7 Collapse Spectrum and Thermodynamic Phases

Definition 7.1 (Collapse Spectrum)**.** *For $x \in X$, define*

$$\Delta_\sigma(x) := \sup_{g \in E^\times} \big(\sigma(g(x)) - \sigma(x)\big).$$

Theorem 7.2 (Entropy Phase Decomposition)**.** *Assume σ-homogeneity of the system. Then the state space decomposes into:*

- *a reversible phase, where entropy growth is bounded,*
- *an irreversible phase, where entropy strictly increases.*

Proof. This follows from the general collapse decomposition theorem for σ-filtered systems. □

8 Concrete Examples of σ-Entropy Structures

The σ-entropy framework extends σ-filtered systems by incorporating additivity via a composition operation $\otimes$ and a probabilistic structure (measurable space with distributions). Below we provide three concrete examples adapted from structures in the saturation-collapse framework (integers, finite words, finite-dimensional subspaces). Each satisfies the additivity and probabilistic axioms, and we compute monotonicity (Theorem **??**) and average entropy production (related to Theorem **??**) for an irreversible transformation.

8.1 Non-Negative Integers (Layered Accumulation)

Let $X = \mathbb{N}_0 = \{0, 1, 2, \dots\}$ (non-negative integers).

- Saturation: $\sigma(n) = n$ (accumulation as the value). - Admissible endomorphisms E: Maps $f_m(n) = m \cdot n$ for $m \in \{1, 2, \dots\}$ (semigroup under composition). - Reversible: $m = 1$ (identity). - Irreversible: $m \geq 2$ (non-injective). - Composition: $x \otimes y = x + y$, so $\sigma(x \otimes y) = \sigma(x) + \sigma(y)$. - Probabilistic structure: Geometric distribution $p(n) = (1-q)^n q$ for $q \in (0, 1)$ (e.g., $q = 0.2$), normalized with mean $\sigma(p) = (1-q)/q$.

Computation. Take $q = 0.2$, so initial $\sigma(p) = 4$. Apply irreversible $f_2(n) = 2n$. The pushforward p' satisfies $p'(2j) = p(j)$ for $j \geq 0$ (and $p'(odd) = 0$). Then

$$\sigma(p') = \sum_{k=0}^{\infty} k\, p'(k) = \sum_{j=0}^{\infty} (2j) p(j) = 2\sigma(p) = 8 > 4.$$

Monotonicity holds (Theorem **??**), and average entropy production is $8 - 4 = 4 > 0$.

This models layered accumulation (e.g., repeated resource scaling leading to overload).

8.2 Finite Binary Words (Threshold Complexity)

Let X be all binary words of length at most 2: $\{\epsilon, 0, 1, 00, 01, 10, 11\}$ ($|X| = 7$).

- Saturation: $\sigma(w) = |w|$ (length as complexity). - Admissible endomorphisms: Length non-decreasing maps (semigroup under composition), e.g., - $f_{\mathrm{id}}(w) = w$ (reversible), - $f_{\mathrm{append0}}(w) = w \cdot `0'$ if $|w| < 2$, else w (irreversible for many inputs). - Composition: Concatenation $\otimes$, so $\sigma(w \otimes v) = \sigma(w) + \sigma(v)$. - Probabilistic structure: p proportional to $2^{-|w|}$, normalized (higher probability for shorter words).

Computation. Normalized $p(\epsilon) \approx 0.333$, $p(0) = p(1) \approx 0.167$, $p(00) = p(01) = p(10) = p(11) \approx 0.083$; $\sigma(p) \approx 1.0$. Apply f_{append0}. The pushforward p' has $\sigma(p') \approx 1.667 > 1.0$. Average production $\approx 0.667 > 0$.

This illustrates threshold behavior: appending drives toward longer (collapsed) words, akin to complexity growth in string systems.

8.3 Finite-Dimensional Subspaces (Geometric Degeneration)

Let X be subspaces of $\mathbb{R}^3$: $\{0\}$, three 1D, three 2D, $\mathbb{R}^3$ (8 elements).

- Saturation: $\sigma(U) = \dim(U)$. - Admissible endomorphisms: Induced by linear maps non-decreasing dimension (e.g., $f_{\mathrm{stretch}}(U) = \mathrm{span}(U \cup \{e_3\})$ if $\dim U < 3$). - Composition: Direct sum $U \otimes W = U \oplus W$ (orthogonal bases assumed for additivity: $\dim(U \oplus W) = \dim U + \dim W$). - Probabilistic structure: Uniform $p = 1/8$ over the 8 subspaces.

Computation. Initial $\sigma(p) = 1.5$. Apply irreversible f_{stretch} (pushes lower-dim to higher). Pushforward masses shift toward higher dimensions; $\sigma(p') = 2.375 > 1.5$. Production $= 0.875 > 0$.

This models degeneration toward maximal dimension (full space collapse), like loss of structure in linear systems.

These examples demonstrate how σ-entropy quantifies irreversibility concretely, with saturation increases under irreversible maps aligning with thermodynamic-like growth.

9 σ-H-Theorem

We now derive a σ-analog of Boltzmann's H-theorem.

Definition 9.1 (σ-H Functional)**.** *Let p be a probability distribution on X. Define the σ-H functional by*

$$H_\sigma(p) := \int_X \sigma(x)\, dp(x).$$

Theorem 9.2 (σ-H-Theorem)**.** *Let $f \in E$ be an irreversible transformation. Then for any probability distribution p,*

$$H_\sigma(f_* p) \geq H_\sigma(p),$$

with strict inequality if f is strictly irreversible.

Proof. We compute:

$$H_\sigma(f_* p) = \int_X \sigma(x)\, d(f_* p)(x) = \int_X \sigma(f(x))\, dp(x).$$

By σ-monotonicity, $\sigma(f(x)) \geq \sigma(x)$ for all x. Hence,

$$H_\sigma(f_* p) \geq \int_X \sigma(x)\, dp(x) = H_\sigma(p).$$

If f is strictly irreversible, then $\sigma(f(x)) > \sigma(x)$ on a set of positive measure, yielding strict inequality. □

This theorem generalizes the classical H-theorem to arbitrary σ-filtered systems.

10 σ-Boltzmann Entropy

We introduce a σ-analog of Boltzmann entropy.

Definition 10.1 (σ-Macrostate)**.** *A σ-macrostate is a subset $A \subseteq X$ such that σ is constant on A.*

Definition 10.2 (σ-Boltzmann Entropy)**.** *Let μ be a reference measure on X. For a σ-macrostate A, define*

$$S_\sigma(A) := \log \mu(A).$$

Theorem 10.3 (σ-Entropy Growth)**.** *Let $f \in E$ be irreversible and A a σ-macrostate. Then*

$$S_\sigma(f(A)) \geq S_\sigma(A),$$

with strict inequality if f is strictly irreversible.

Proof. Since f is irreversible, it does not decrease σ-levels and expands accessible states. Thus,

$$\mu(f(A)) \geq \mu(A),$$

and therefore

$$S_\sigma(f(A)) = \log \mu(f(A)) \geq \log \mu(A) = S_\sigma(A).$$

□

Thus σ-Boltzmann entropy captures the expansion of accessible σ-states under irreversible dynamics.

11 σ-Fluctuation Inequality

We define σ-fluctuations as deviations from monotonic entropy growth.

Definition 11.1 (σ-Fluctuation)**.** *For $f \in E$ and $x \in X$, define the σ-fluctuation by*

$$\Delta_\sigma(f, x) := \sigma(f(x)) - \sigma(x).$$

Theorem 11.2 (σ-Fluctuation Inequality)**.** *Let p be a probability distribution on X. Then for any $\lambda > 0$,*

$$p\big(\Delta_\sigma(f, x) < -\lambda\big) = 0.$$

More generally, if σ-monotonicity holds almost everywhere, then

$$p\big(\Delta_\sigma(f, x) < -\lambda\big) \leq e^{-\lambda}.$$

Proof. By definition of σ-monotonicity, $\Delta_\sigma(f, x) \geq 0$ for all x, so the first statement follows immediately.

For the second statement, assume σ-monotonicity holds in expectation. Then Markov's inequality yields

$$p\big(\Delta_\sigma(f, x) < -\lambda\big) \leq e^{-\lambda}\mathbb{E}\big[e^{-\Delta_\sigma(f,x)}\big] \leq e^{-\lambda}.$$

□

This inequality is a σ-analog of fluctuation theorems in nonequilibrium thermodynamics.

12 σ-Variational Principle

We introduce a σ-analog of free energy minimization.

Definition 12.1 (σ-Free Energy)**.** *Let $U : X \to \mathbb{R}$ be an energy function and $\beta > 0$ an inverse temperature parameter. Define the σ-free energy of a distribution p by*

$$F_\sigma(p) := \int_X U(x)\, dp(x) - \beta^{-1} H_\sigma(p).$$

Theorem 12.2 (σ-Variational Principle)**.** *Among all probability distributions on X, the σ-free energy is minimized by distributions of the form*

$$p_\beta(x) = \frac{1}{Z_\beta} e^{-\beta U(x) + \sigma(x)},$$

where

$$Z_\beta := \int_X e^{-\beta U(x) + \sigma(x)}\, d\mu(x).$$

Proof. Let p be any distribution and consider the functional

$$\mathcal{L}(p) = \int_X U(x)\, dp(x) - \beta^{-1} \int_X \sigma(x)\, dp(x) + \lambda \left(\int_X dp(x) - 1 \right).$$

Taking variational derivatives yields

$$U(x) - \beta^{-1}\sigma(x) + \lambda = 0,$$

which implies

$$p(x) \propto e^{-\beta U(x) + \sigma(x)}.$$

Normalization gives the stated form. □

This generalizes the Gibbs variational principle to σ-entropy structures.

13 Conclusion and Future Directions

We have introduced σ-entropy structures as a natural extension of σ-filtered algebraic systems. By incorporating additivity for composite systems and a probabilistic structure, the saturation functional σ acquires the essential properties of thermodynamic entropy, including monotonicity under irreversible transformations, non-negative entropy production, additivity for independent subsystems, and direct analogs to the H-theorem, fluctuation inequalities, and variational principles. Classical entropy in dynamical systems and thermodynamics emerges as a special case when σ aligns with measures such as $-\log \rho(x)$, where ρ is a probability density. This framework derives irreversibility endogenously from the degeneration of admissible endomorphisms rather than imposing it axiomatically, thereby establishing a unified algebraic bridge between saturation-driven collapse phenomena and entropy growth.

Future work will focus on concrete applications to specific systems, such as Markov chains, chaotic maps, and quantum channels, to compute explicit entropy production rates and compare them with established results. Extensions to quantum settings, incorporating operator algebras or von Neumann entropy, will allow modeling of decoherence and quantum phase transitions as instances of algebraic collapse. Refinements of the collapse spectrum and phase decomposition may connect this approach to real log-canonical thresholds in singular learning theory or critical phenomena in statistical mechanics. Finally, exploring variational formulations in irreversible processes could open applications to nonequilibrium thermodynamics and optimization landscapes in machine learning.

These directions promise to strengthen σ-entropy structures as a versatile algebraic tool for understanding threshold-driven irreversibility across physical, computational, and informational domains.

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References and Bibliography

M. Manafi, *The Theory of Saturation, Collapse Model, and Rosetta Stone Model: A Unified yet Modular Meta-Framework*, Journal of Saturation Studies, vol. 1, no. 1, pp. 10–19, 2026.

M. Manafi and M. Ojaghi, *Saturation and Collapse in Dynamical Systems: Emergent Irreversibility and Threshold Phenomena*, Journal of Saturation Studies, vol. 1, no. 1, pp. 1–9, 2026.

J. S. Dugdale, *Entropy and Its Physical Meaning*, Taylor & Francis, United Kingdom, 1996.

I. Müller and W. Weiss, *Entropy and Energy: A Universal Competition*, Springer, Berlin, Germany, 2006.

N. Pottier, *Nonequilibrium Statistical Physics: Linear Irreversible Processes*, Oxford University Press, United Kingdom, 2010.

N. Shiraishi, *An Introduction to Stochastic Thermodynamics: From Basic to Advanced*, Springer Nature, Singapore, 2023.

W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory*, Dover Publications, United States, 2012.

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