

Collapse Structures: An Algebraic Framework for Irreversible Dynamics

Mahmoud Manafi

Abstract

This paper introduces collapse structures as a minimal algebraic framework that captures the distinction between reversible and irreversible dynamics through a monotone invariant. In addition, it is tried to define the basic objects, introduce key notions (rigidity, σ -homogeneity, σ -collapse spectrum), prove fundamental results (invariance and sharp threshold theorems), and finally construct the natural category of collapse structures together with compatible morphisms.

1 Introduction

In the research conducted by Manafi and Ojaghi (2026), novel perspectives associated with the concept of saturation and collapse were proposed. Building on their work, several algebraic structures have been developed regarding geometry, category theory, and dynamical systems.

First, the concept of a lattice is replaced by the set of real numbers in order to extend the notion introduced by Manafi and Ojaghi. This approach enables discussion of various topics in algebraic geometry and dynamical systems. In addition, the collapse structure category is defined for the first time, providing a deeper understanding and a framework for exploring new structures. Throughout this paper, some theorems related to this topic are proved, thereby facilitating further research and systematic development of the subject.

2 Basic Definitions

Let Γ be a totally ordered commutative monoid $(\Gamma, \leq, +)$ with neutral element 0_Γ (examples include $(\mathbb{R}_{\geq 0}, \leq, +)$, $(\mathbb{Z}_{\geq 0}, \leq, +)$, $([0, \infty], \leq, \min)$, etc.).

Definition 2.1 (Collapse Structure). *A collapse structure is a triple*

$$S = (S, \sigma, G)$$

consisting of

- S a nonempty set (called the state space),
- $\sigma : S \rightarrow \Gamma$ a function (called the collapse level or irreversibility measure),
- $G \subseteq \{f : S \rightarrow S\}$ a subsemigroup under function composition (called the dynamics semigroup)

such that

$$\forall g \in G, \forall x \in S, \sigma(g(x)) \geq \sigma(x).$$

The condition $\sigma(g(x)) \geq \sigma(x)$ expresses that every admissible transformation is non-decreasing with respect to the irreversibility measure σ .

Definition 2.2 (Reversible part of the dynamics). *The reversible dynamics of S are given by*

$$G^\times := \{g \in G \mid g \text{ is bijective and } g^{-1} \in G\}.$$

Elements of G^\times are called reversible transformations; elements of $G \setminus G^\times$ are called irreversible.

Definition 2.3 (Rigid states). *A state $x \in S$ is called rigid if*

$$\forall g \in G^\times, \sigma(g(x)) = \sigma(x).$$

Let

$$R(S) := \{x \in S \mid x \text{ is rigid}\}.$$

States not in $R(S)$ are called non-rigid.

3 Key Algebraic Properties

Definition 3.1 (σ -Homogeneity). *A collapse structure $S = (S, \sigma, G)$ is σ -homogeneous if*

$$\forall x, y \in S, \sigma(x) = \sigma(y) \Rightarrow \exists g \in G^\times \text{ such that } g(x) = y.$$

Definition 3.2 (σ -Collapse Spectrum). *The σ -collapse spectrum of S is the set*

$$\text{Spec}_\sigma(S) := \{\alpha \in \Gamma \mid \exists x \in S \setminus R(S) \text{ with } \sigma(x) = \alpha\}.$$

The collapse threshold is defined as

$$\alpha_c(S) := \sup \text{Spec}_\sigma(S) \in \Gamma \cup \{+\infty\}.$$

Theorem 3.1 (Sharp Threshold Theorem). *If S is σ -homogeneous, then*

$$\text{Spec}_\sigma(S) = \{\alpha \in \Gamma \mid \alpha < \alpha_c(S)\}.$$

Proof sketch. Downward closure: suppose $\alpha \in \text{Spec}_\sigma(S)$ and $\beta < \alpha$. Let x witness α (i.e. $\sigma(x) = \alpha$ and x non-rigid). Let y be any state with $\sigma(y) = \beta$. By σ -homogeneity there exists $h \in G^\times$ with $h(y) = x'$, $\sigma(x') = \alpha$. Then $g \circ h(y)$ for suitable g that increases σ at x' gives irreversible growth at level β .

4 The Category of Collapse Structures

Definition 4.1 (σ -Morphism). *Let $S_1 = (S_1, \sigma_1, G_1)$ and $S_2 = (S_2, \sigma_2, G_2)$ be collapse structures.*

A σ -morphism

$$\phi : S_1 \rightarrow S_2$$

is a function $\phi : S_1 \rightarrow S_2$ satisfying:

1. *Collapse level monotonicity:*

$$\sigma_2(\phi(x)) \geq \sigma_1(x)$$

2. *Dynamics compatibility: for every $g \in G_1$ there exists $h \in G_2$ such that*

$$\phi \circ g = h \circ \phi$$

Definition 4.2 (The category Coll). *We denote by Coll the category whose*

- *objects are collapse structures (S, σ, G)*
- *morphisms are σ -morphisms*
- *composition is ordinary function composition*

When the target monoid Γ is fixed across objects one writes Coll_Γ . If we further restrict to morphisms preserving σ exactly ($\sigma_2(\phi(x)) = \sigma_1(x)$) we obtain a subcategory Coll_Γ^b .

Definition 4.3 (σ -Equivalence). *Two collapse structures S_1 and S_2 are σ -equivalent if there exists a bijective σ -morphism $\phi : S_1 \rightarrow S_2$ whose inverse is also a σ -morphism.*

Theorem 4.1 (Invariance of the Collapse Spectrum). *If $S_1 \cong S_2$ in Coll then*

$$\text{Spec}_\sigma(S_1) = \text{Spec}_\sigma(S_2)$$

and

$$\alpha_c(S_1) = \alpha_c(S_2).$$

Every object of Coll can be viewed as a graded set

$$S = \bigsqcup_{\alpha \in \Gamma} \sigma^{-1}(\alpha)$$

together with a semigroup G acting in a monotone way on the grading.

5 Obstruction to Reversibility Induced by Monotone Collapse Invariants

5.1 Setup

Let (S, σ, G) be a collapse structure where

- S is a nonempty set
- G is a semigroup acting on S
- $\sigma : S \rightarrow \mathbb{R}$ satisfies

$$\sigma(g \cdot x) \leq \sigma(x)$$

and is nontrivial.

5.2 Reversibility and Extensions

Let H be a group acting on S . We say H extends G if

- $G \subseteq H$
- the restriction of the H action to G coincides with the given action

5.3 Obstruction Theorem

Theorem 5.1 (Obstruction to Reversibility). *Let (S, σ, G) be a collapse structure such that*

1. σ is monotone with respect to G
2. there exist $g \in G$ and $x \in S$ such that

$$\sigma(g \cdot x) < \sigma(x)$$

Then there exists no group H acting on S extending G and preserving σ .

Proof. Assume such a group H exists. Since $g \in H$, its inverse g^{-1} exists.

Because σ is invariant under H we have

$$\sigma(g \cdot x) = \sigma(x).$$

But by assumption

$$\sigma(g \cdot x) < \sigma(x).$$

Applying g^{-1} yields

$$\sigma(x) = \sigma(g^{-1}(g \cdot x)) = \sigma(g \cdot x),$$

a contradiction. □

6 Conclusion

We developed the framework of collapse structures as a minimal algebraic model for irreversibility based on monotone invariants. By introducing key notions such as σ -homogeneity, rigid states, the σ -collapse spectrum, and the collapse threshold, we clarified the structural distinction between reversible and irreversible dynamics.

The categorical formulation and obstruction theorem demonstrate that irreversibility is not an ad hoc assumption but a necessary consequence of strictly monotone invariants in algebraic and dynamical systems.

Future work may explore collapse structures in richer algebraic and geometric contexts, deeper categorical properties, and applications to dynamical systems and algebraic geometry.

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manafi@manafi-institute.de